Problem 1 Domains

a. Let $D_1$ and $D_2$ be CPOs.
   i. Prove that any continuous function from $D_1$ to $D_2$ is monotonic.
   ii. The componentwise ordering of $D_1 \times D_2$ is the relation $\sqsubseteq_{D_1 \times D_2}$ defined by:

   $$(d_1, d_2) \sqsubseteq_{D_1 \times D_2} (d'_1, d'_2) \text{ if and only if } d_1 \sqsubseteq_{D_1} d'_1 \text{ and } d_2 \sqsubseteq_{D_2} d'_2$$

   Prove that $D_1 \times D_2$ is a CPO under the componentwise ordering.

b. Let $D$ be the following domain:

   $\begin{array}{cccc}
   a & b & c & d \\
   \downarrow & & & \\
   \bot & & & 
   \end{array}$

   Let $T$ be the set of total functions from $D$ to $D$, let $M$ be the set of monotonic functions from $D$ to $D$, and let $C$ be the set of continuous functions from $D$ to $D$.

   i. What is the size of $T$?
   ii. What is the size of $M$? Give the graph of a function in $(T - M)$, if any.
   iii. What is the size of $C$? Give the graph of a function in $(M - C)$, if any.

c. (This part (c) of Problem #1 is OPTIONAL. It will be graded for extra-credit on this problem set, but its completion is not required.)

   Let $D$ be a pointed CPO, and let $(D \to D)$ be the CPO of continuous functions from $D$ to $D$, ordered pointwise (this is the ordering defined on page 181 of the course notes, “Function Domains”).

   Define a function $Y$ from $(D \to D)$ to $D$ as follows:

   $$Y(f) = \bigsqcup_{n \geq 0} f^n(\bot_D)$$

   Prove that $Y$ is a continuous function.

Addendum:
You may use the following lemmas in your proof.
Lemma 1 1 Let $D$ be a pointed CPO, let $n$ be any non-negative integer, and let $F$ be a chain of functions in $(D \to D)$. Then:

$$\left(\bigsqcup_{f \in F} f^n(\bot_D)\right) = \bigsqcup_{f \in F} f^n(\bot_D)$$

Lemma 2 2 Let $D$ be a pointed CPO and let $F$ be a chain of functions in $(D \to D)$. Then:

$$\bigsqcup_{n \geq 0} \left(\bigsqcup_{f \in F} f^n(\bot_D)\right) = \bigsqcup_{f \in F} \left(\bigsqcup_{n \geq 0} f^n(\bot_D)\right)$$

Problem 2 (Substitution)

Do exercise 6.16 (on page 238).

Problem 3 (Operational semantics of FLK)

Louis Reasoner has an idea for a new FLK command, $(\text{terminate } E_1 \ E_2)$. If either $E_1$ or $E_2$ terminates with a value or an error, $(\text{terminate } E_1 \ E_2)$ will also terminate with a value or an error. In evaluating terminate, we run one expression for one step, then run the other expression for one step, and so on. Louis initially worked out a few examples of how his new construct would work:

$$(\text{terminate } 1 \ 2) \Rightarrow 1 \text{ or } 2 \text{ (implementation dependent)}$$
$$(\text{terminate } 2 \ (\text{rec } x \ x)) \Rightarrow 2$$
$$(\text{terminate } 1 \ (\text{call } 3 \ 0)) \Rightarrow 1 \text{ or error: can’t apply non-procedure (implementation dependent)}$$
$$(\text{terminate } (\text{rec } x \ x) \ (/ \ 3 \ 0)) \Rightarrow \text{error: divide by zero}$$
$$(\text{terminate } (\text{rec } x \ x) \ (\text{rec } x \ x)) \Rightarrow \bot$$

Louis is really excited about the terminate construct. His old implementation of FLK required him to reboot any time his program ran into an infinite loop. Although he hasn’t solved the halting problem, now he can guarantee not to have to reboot (excepting, of course, when his new-fangled operating system crashes) by testing his programs with terminate and his new $(\text{timer } N)$ construct.

Louis defined the following reduction rule(s) for timer:

$$(\text{timer } N_1) \rightsquigarrow (\text{timer } N_2)$$

where $N_1 > 1$ and $N_2 = N_1 - 1$  \hspace{1cm} \text{[timer-countdown]}

$$(\text{timer } 1) \rightsquigarrow \#u$$

$$(\text{terminate } (\text{timer } 1000) \ \text{might-go-infinite})$$
Unfortunately, Louis set off for Hawaii before he was able to extend the FL Operational Semantics to include \texttt{terminate}. In his absence, you are asked to finish it up.

a. Add new reduction rules ($\rightarrow$) and new evaluation progress rules ($\Rightarrow$) for \texttt{terminate}.

b. Are your rules confluent?

c. Show how the following expression would be evaluated using the rules above:

\[
(\texttt{terminate} \ (\texttt{call} \ (\texttt{proc} \ x \ (\texttt{primop} \ + \ x \ 2)) \ 5) \\
(\texttt{if} \ (> \ 4 \ 3) \\
\quad (\texttt{rec} \ x \ x) \\
\quad (\texttt{proc} \ y \ 1)))
\]